Journal of Computational and Applied Mathematics 235 (2011) 2372–2379

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Positive solutions for three-point boundary value problems for second-order impulsive differential equations on infinite intervals[☆]

Jianli Li[a,*], Jianhua Shen[b]

[a] Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China

[b] Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China

ARTICLE INFO

Article history:
Received 2 September 2009
Received in revised form 13 September 2010

MSC:
34A37
34K10

Keywords:
Three-point boundary value problem
Impulses
Infinite interval
Diagonalization process

ABSTRACT

This paper discusses the existence of positive solutions for three-point boundary value problems for second-order nonlinear impulsive differential equations on an infinite interval. By using the diagonalization process, fixed point theory and inequality techniques, sufficient conditions for guaranteeing the existence of positive solutions are obtained.

1. Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years. For some general and recent works on the theory of impulsive differential equations we refer the reader to [1–3]. Applications of impulsive differential equations occur in biology, medicine, mechanics, engineering, etc. [4–12]. But the corresponding theory for impulsive boundary value problems on an infinite interval has yet to be fully developed; we have only seen [13–15].

In this paper, we shall use the diagonalization process and the Leray–Schauder fixed point theory to investigate the existence of positive solutions for the following boundary value problem:

$$\begin{aligned} &x''(t)+\phi(t)f(t,x(t),x'(t)), \quad t\neq t_k,\\ &\Delta x'(t_k)=b_k x'(t_k), \qquad \Delta x(t_k)=a_k x(t_k), \quad k=1,2,\ldots \\ &x(0)=\alpha x(\eta), \qquad x'(+\infty)=0, \end{aligned} \tag{1.1}$$

where $0=t_0<t_1<t_2<\cdots<t_k<\cdots$, $\lim_{k\to+\infty}t_k=+\infty$, $\Delta x'(t_k)=x'(t_k^+)-x'(t_k^-)$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, $0<\alpha<1$, $\eta>0$, $x'(+\infty)=\lim_{t\to+\infty}x'(t)$.

☆ This work was supported by the NNSF of China (No. 10871062), a project supported by Hunan Provincial Natural Science Foundation of China (No. 10JJ6002) and the Zhejiang Provincial Natural Science Foundation of China (No. Y6090057).

* Corresponding author.
E-mail address: ljianli@sina.com (J. Li).

0377-0427/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2010.10.037

Let $J = [0, a]$, where a is a constant or $a = +\infty$; we introduce the following spaces of functions:
$PC(J) = \{u : J \to R, u$ is continuous at $t \neq t_k$, $u(t_k^+)$, $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k)\}$.
$PC^1(J) = \{u \in PC(J) : u$ is continuously differentiable at $t \neq t_k$, $u'(0^+)$, $u'(t_k^+)$, $u'(t_k^-)$ exist and $u'(t_k^-) = u'(t_k)\}$.
$PC^2(J) = \{u \in PC^1(J) : u$ is twice continuously differentiable at $t \neq t_k\}$.
Note that $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|u\|_\infty = \sup\{|u(t)| : t \in J\}, \quad \text{and} \quad \|u\|_1 = \max\{\|u\|_\infty, \|u'\|_\infty\}$$

respectively.

2. Preliminaries

Definition 2.1. The set $\mathcal{F}$ is said to be quasiequicontinuous in $[0, c]$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in \mathcal{F}$, $k \in Z$, $t^*, t^{**} \in (t_{k-1}, t_k] \cap [0, c]$ and $|t^* - t^{**}| < \delta$, then $|x(t^*) - x(t^{**})| < \varepsilon$.

Lemma 2.1 (*[2], Compactness Criterion*). *The set $\mathcal{F} \subset PC([0, c], R^n)(c < +\infty)$ is relatively compact if and only if*

1. *$\mathcal{F}$ is bounded;*
2. *$\mathcal{F}$ is quasiequicontinuous in $[0, c]$.*

Lemma 2.2 (*[16], Nonlinear Alternative of Leray–Schauder Type*). *Let E be a Banach space and $V \subseteq E$ a convex set. Assume Ω is a relatively open subset of V with $p^* \in \Omega$ and $S : \bar{\Omega} \to V$ a continuous, compact map. Then either*

(i) *S has a fixed point or*
(ii) *there exists $u \in \partial\Omega$ and $\lambda \in (0, 1)$ with $u = \lambda Su + (1 - \lambda)p^*$.*

Consider the boundary value problems on finite intervals for the problem

$$\begin{cases} x'' + \phi(t)F(t, x, x') = 0, \quad 0 < t < n, t \neq t_k, \\ \Delta x'(t_k) = b_k x'(t_k), \qquad \Delta x(t_k) = a_k x(t_k), \quad k = 1, 2, \ldots \\ x(0) = \alpha x(\eta), \qquad x'(n) = b, \end{cases} \tag{2.1}$$

where $n > \eta$ is a any real number.

Lemma 2.3. *Suppose $\phi \in C(0, n)$ with $\phi > 0$ on $(0, n)$ and $\phi \in L^1[0, n]$ and $F : [0, n] \times R \times R \to R$ is continuous. Further suppose there is a constant $M > \left(\frac{\alpha\eta}{1-\alpha} + n\right) b$, independent of λ, with $\|x\|_1 = \max\{\|x\|, \|x'\|\} \neq M$, where $\|y\| = \sup_{0 \le t \le n} |y(t)|$ if $y = x$ or x', for any solution $x \in PC^1[0, n] \cap PC^2(0, n)$ to*

$$\begin{cases} x'' + \lambda\phi(t)F(t, x, x') = 0, \quad 0 < t < n, t \neq t_k, \\ \Delta x'(t_k) = \lambda b_k x'(t_k), \qquad \Delta x(t_k) = \lambda a_k x(t_k), \quad k = 1, 2, \ldots, \\ x(0) = \alpha x(\eta), \qquad x'(n) = b \end{cases} \tag{2.1$^\lambda$}$$

for each $\lambda \in (0, 1)$. Then (2.1) has a solution $x \in PC^1[0, n] \cap PC^2(0, n)$ with $\|x\|_1 \le M$.

Proof. (2.1) has a solution if and only if the operator S:

$$\begin{aligned} Sx(t) = {} & \left[\frac{\alpha\eta}{1-\alpha} + t\right] b + \frac{\alpha}{1-\alpha} \int_0^\eta \int_s^n \phi(\tau) f(\tau, x, x') \mathrm{d}\tau \mathrm{d}s - \frac{\alpha}{1-\alpha} \sum_{0<t_k<n} b_k x'(t_k)\eta \\ & + \frac{\alpha}{1-\alpha} \left[\sum_{0<t_k\le\eta} b_k x'(t_k)(\eta - t_k) + \sum_{0<t_k<\eta} a_k x(t_k)\right] + \int_0^t \int_s^n \phi(\tau) f(\tau, x, x') \mathrm{d}\tau \mathrm{d}s \\ & - \sum_{0<t_k<n} b_k x'(t_k) t + \sum_{0<t_k\le t} b_k x'(t_k)(t - t_k) + \sum_{0<t_k<t} a_k x(t_k) \end{aligned}$$

has a fixed point. It is easily shown that S is continuous and compact. Let $w(t) = \left(\frac{\alpha\eta}{1-\alpha} + t\right) b$. Then (2.1$^\lambda$) has a solution x if and only if $x = (1 - \lambda)w + \lambda Sx$. Let $\Omega = \{u \in PC^1[0, n] : \|u\|_1 < M\}$, $V = E = PC^1[0, n]$. For $x \in \partial\Omega$, $x \neq (1 - \lambda)w + \lambda Sx$. Lemma 2.2 guarantees that (2.1) has a solution x with $\|x\|_1 \le M$. This proof is complete. □

Lemma 2.4. *If x satisfies (2.1$^\lambda$), then*

$$x'(t) = \prod_{t<t_k<n} (1 + \lambda b_k)^{-1} x'(n) + \lambda \int_t^n \prod_{t<t_k\le s} (1 + \lambda b_k)^{-1} \phi(s) F(s, x(s), x'(s)) \mathrm{d}s. \tag{2.2}$$

Proof. We assume, without loss of generality, that $t \in (0, t_1)$, $n \in (t_2, t_3]$. From (2.1$^\lambda$), we have

$$x'(n) - x'(t_2^+) = -\lambda \int_{t_2}^{n} \phi(s)F(s, x(s), x'(s))\mathrm{d}s,$$
$$x'(t_2) - x'(t_1^+) = -\lambda \int_{t_1}^{t_2} \phi(s)F(s, x(s), x'(s))\mathrm{d}s,$$
$$x'(t_1) - x'(t) = -\lambda \int_{t}^{t_1} \phi(s)F(s, x(s), x'(s))\mathrm{d}s.$$

From $x'(t_k) = (1 + \lambda b_k)^{-1}x'(t_k^+)$, we obtain

$$\begin{aligned}
x'(t) &= x'(t_1) + \lambda \int_t^{t_1} \phi(s)F(s, x(s), x'(s))\mathrm{d}s \\
&= (1 + \lambda b_1)^{-1}x'(t_1^+) + \lambda \int_t^{t_1} \phi(s)F(s, x(s), x'(s))\mathrm{d}s \\
&= (1 + \lambda b_1)^{-1}\left[x'(t_2) + \lambda \int_{t_1}^{t_2} \phi(s)F(s, x(s), x'(s))\mathrm{d}s\right] + \lambda \int_t^{t_1} \phi(s)F(s, x(s), x'(s))\mathrm{d}s \\
&= (1 + \lambda b_1)^{-1}(1 + \lambda b_2)^{-1}x(t_2^+) + \lambda \int_{t_1}^{t_2} (1 + \lambda b_1)^{-1}\phi(s)F(s, x(s), x'(s))\mathrm{d}s + \lambda \int_t^{t_1} \phi(s)F(s, x(s), x'(s))\mathrm{d}s \\
&= (1 + \lambda b_1)^{-1}(1 + \lambda b_2)^{-1}x'(n) + \lambda \int_{t_2}^{n} (1 + \lambda b_1)^{-1}(1 + \lambda b_2)^{-1}\phi(s)F(s, x(s), x'(s))\mathrm{d}s \\
&\quad + \lambda \int_{t_1}^{t_2} (1 + \lambda b_1)^{-1}\phi(s)F(s, x(s), x'(s))\mathrm{d}s + \lambda \int_t^{t_1} \phi(s)F(s, x(s), x'(s))\mathrm{d}s \\
&= \prod_{t<t_k<n} (1 + \lambda b_k)^{-1}x'(n) + \lambda \int_t^n \prod_{t<t_k\le s} (1 + \lambda b_k)^{-1}\phi(s)F(s, x(s), x'(s))\mathrm{d}s,
\end{aligned}$$

which completes the proof. □

3. Main results

Theorem 3.1. *Suppose the following hold.*

(H$_1$) $\phi \in C(0, +\infty)$ *with* $\phi > 0$ *on* $(0, +\infty)$ *and* $Q_\infty = \int_0^{+\infty} \prod_{0<t_k<s}(1+b_k^-)^{-1}\phi(s)\mathrm{d}s < +\infty$, $R_\infty = \int_0^{+\infty} s\phi(s)\mathrm{d}s < +\infty$. $a_k > -1$, $b_k > -1$, *and* $\prod_{k=1}^{\infty}(1 + a_k^+) = A < \frac{1}{\alpha}$, $\prod_{k=1}^{\infty}(1 + b_k^+)^{-1} = B_1 < +\infty$, $\prod_{k=1}^{\infty}(1 + b_k^-)^{-1} = B < +\infty$, *where* $s^+ = \max\{s, 0\}$, $s^- = \min\{s, 0\}$, $s = a_k$ *or* b_k.

(H$_2$) $f : [0, +\infty) \times [0, +\infty) \times (0, +\infty) \to (0, +\infty)$ *is continuous with* $f(t, x, v) > 0$ *for* $(t, x, v) \in [0, +\infty) \times (0, +\infty) \times (0, +\infty)$.

(H$_3$) $f(t, x, v) \le \omega(\max\{x, v\})$ *on* $[0, +\infty) \times (0, +\infty) \times (0, +\infty)$ *with* $\omega \ge 0$ *continuous and nondecreasing on* $[0, +\infty)$.

(H$_4$)
$$\sup_{c\in(0,+\infty)} \frac{c}{\omega(c) \max\left\{Q_\infty, \frac{\alpha A^2 B}{1-\alpha A} \int_0^\eta s\phi(s)\mathrm{d}s + ABR_\infty\right\}} > 1.$$

(H$_5$) *For a constant* $H > 0$ *there exists a function* ψ_H *continuous on* $[0, +\infty)$ *and positive on* $(0, +\infty)$ *and a constant* γ, $0 \le \gamma < 1$ *with* $f(t, x, v) \ge \psi_H(t)v^\gamma$ *on* $[0, +\infty) \times [0, H]^2$.

Then the problem (1.1) *has solutions* $x \in PC^1[0, +\infty) \cap PC^2(0, +\infty)$ *with* $x > 0$ *on* $(0, +\infty)$.

Proof. First fix $n \in N_\eta = \{[\eta] + 1, [\eta] + 2, \dots, \}$, where $[\cdot]$ is an integral function, and consider the family problems

$$\begin{aligned}
&x'' + \phi(t)f(t, x, x') = 0, \quad 0 < t < n,\ t \ne t_k, \\
&\Delta x'(t_k) = b_k x'(t_k), \qquad \Delta x(t_k) = a_k x(t_k), \\
&x(0) = \alpha x(\eta), \qquad x'(n) = 0.
\end{aligned} \tag{3.1}$$

Choose $M > 0$ with

$$\frac{M}{\omega(M) \max\left\{Q_\infty, \frac{\alpha A^2 B}{1-\alpha A} \int_0^\eta s\phi(s)\mathrm{d}s + ABR_\infty\right\}} > 1.$$

Next choose $\varepsilon > 0$ with $\varepsilon < M$, and

$$\frac{M}{\omega(M)\max\left\{Q_\infty, \frac{\alpha A^2 B}{1-\alpha A}\int_0^\eta s\phi(s)\mathrm{d}s + ABR_\infty\right\} + \varepsilon} > 1. \tag{3.2}$$

Let $n_0 \in N \setminus \{0\}$ be chosen such that $\frac{1}{n_0}\left(\frac{\alpha\eta A^2 B}{1-\alpha A} + nAB\right) < \varepsilon$; we first show that

$$\begin{aligned} &x'' + \phi(t)f^*(t, x, x') = 0, \quad 0 < t < n,\ t \neq t_k,\\ &\Delta x'(t_k) = b_k x'(t_k), \qquad \Delta x(t_k) = a_k x(t_k), \quad k = 1, 2, \ldots\\ &x(0) = \alpha x(\eta), \qquad x'(n) = \frac{1}{m} \end{aligned} \tag{3.3^m}$$

has a solution for each $m \in \{n_0, n_0 + 1, \ldots\}$, where

$$f^*(t, x, v) = \begin{cases} f(t, x, v), & x \geq 0,\ v \geq \frac{B_1}{m},\\ f\left(t, x, \frac{B_1}{m}\right), & x \geq 0,\ v \leq \frac{B_1}{m}\\ f(t, 0, v), & x < 0,\ v \geq \frac{B_1}{m}\\ f\left(t, 0, \frac{B_1}{m}\right), & x < 0,\ v \leq \frac{B_1}{m}. \end{cases}$$

To show that (3.3^m) has a solution, we consider the family of problems

$$\begin{aligned} &x'' + \lambda\phi(t)f^*(t, x, x') = 0, \quad 0 < t < n,\ t \neq t_k,\\ &\Delta x'(t_k) = \lambda b_k x'(t_k), \qquad \Delta x(t_k) = \lambda a_k x(t_k), \quad k = 1, 2, \ldots\\ &x(0) = \alpha x(\eta), \qquad x'(n) = \frac{1}{m} \end{aligned} \tag{3.4^m_λ}$$

for $0 < \lambda < 1$. Let $x \in PC^1[0, n] \cap PC^2(0, n)$ be any solution of (3.4^m_λ). It follows from (H_2) and Lemma 2.4 that

$$\begin{aligned} x'(t) &= \prod_{t<t_k<n}(1 + \lambda b_k)^{-1}x'(n) + \lambda\int_t^n \prod_{t<t_k\leq s}(1 + \lambda b_k)^{-1}\phi(s)f^*(s, x(s), x'(s))\mathrm{d}s\\ &\leq \prod_{t<t_k<n}(1 + b_k^-)^{-1}x'(n) + \int_t^n \prod_{t<t_k\leq s}(1 + b_k^-)^{-1}\phi(s)f^*(s, x(s), x'(s))\mathrm{d}s\\ &\leq \frac{B}{m} + \omega(\|x\|_1)\int_t^n \prod_{t<t_k\leq s}(1 + b_k^-)^{-1}\phi(s)\mathrm{d}s. \end{aligned} \tag{3.5}$$

From $\frac{1}{n_0}\left(\frac{\alpha A^2 B}{1-\alpha A} + nAB\right) < \varepsilon$, $A \geq 1$ and $m \in \{n_0, n_0 + 1, \ldots\}$, we have $\frac{B}{m} < \varepsilon$. Hence

$$x'(t) \leq \varepsilon + \omega(\|x\|_1)\int_t^n \prod_{t<t_k\leq s}(1 + b_k)^{-1}\phi(s)\mathrm{d}s \leq \varepsilon + \omega(\|x\|_1)Q_\infty. \tag{3.6}$$

From (2.2) of Lemma 2.4, we have $x'(t) \geq \frac{B_1}{m}$, so

$$x(t) \geq \prod_{0<t_k<t}(1 + a_k)x(0) + \int_0^t \prod_{0<t_k<s}(1 + a_k)\frac{B_1}{m}\mathrm{d}s.$$

This and $x(0) = \alpha x(\eta)$ yield

$$x(0) \geq \frac{\alpha B_1}{m\left(1 - \alpha\prod_{0<t_k<\eta}(1 + a_k)\right)}\int_0^\eta \prod_{0<t_k<s}(1 + a_k)\mathrm{d}s > 0,$$

and thus $x(t) > 0$. From (3.5) and $x(t_k^+) = (1 + a_k)x(t_k)$ we obtain that

$$\begin{aligned} x(t) &\le x(0) \prod_{0<t_k<t} (1+a_k) + \int_0^t \prod_{0<t_k<s} (1+a_k) \left[\frac{B}{m} + \omega(\|x\|_1) \int_s^n \prod_{s<t_k\le\tau} (1+b_k^-)^{-1} \phi(\tau)\mathrm{d}\tau \right] \mathrm{d}s \\ &\le Ax(0) + \frac{AB}{m} t + AB\omega(\|x\|_1) \int_0^t s\phi(s)\mathrm{d}s, \quad t \in [0, n]. \end{aligned}$$

This together with $x(0) = \alpha x(\eta)$ yields

$$\begin{aligned} x(t) &\le \frac{\alpha A}{1-\alpha A} \left[\frac{AB\eta}{m} + AB\omega(\|x\|_1) \int_0^\eta s\phi(s)\mathrm{d}s \right] + \frac{AB}{m} t + AB\omega(\|x\|_1) \int_0^t s\phi(s)\mathrm{d}s \\ &\le \frac{\alpha\eta}{1-\alpha A} \frac{A^2B}{m} + \frac{n}{m} AB + \omega(\|x\|_1) \left[\frac{\alpha A^2 B}{1-\alpha A} \int_0^\eta s\phi(s)\mathrm{d}s + AB \int_0^{+\infty} s\phi(s)\mathrm{d}s \right]. \end{aligned}$$

From $\frac{1}{n_0}(\frac{\alpha A^2 B}{1-\alpha A} + nAB) < \varepsilon$ and $m \in \{n_0, n_0 + 1, \ldots\}$, we have

$$x(t) \le \varepsilon + \omega(\|x\|_1) \left[\frac{\alpha A^2 B}{1-\alpha A} \int_0^\eta s\phi(s)\mathrm{d}s + ABR_\infty \right]. \tag{3.7}$$

We combine (3.6) and (3.7) and find

$$\frac{\|x\|_1}{\omega(\|x\|_1) \max\left\{Q_\infty, \frac{\alpha A^2 B}{1-\alpha A} \int_0^\eta s\phi(s)\mathrm{d}s + ABR_\infty\right\} + \varepsilon} \le 1. \tag{3.8}$$

Now (3.2) together with (3.8) implies $\|x\|_1 \ne M$. Thus Lemma 2.3 implies that (3.3^m) has a solution $x_{m,n}$ with $\|x_{m,n}\|_1 \le M$. In fact

$$0 < x_{m,n}(t) \le M, \qquad \frac{B_1}{m} \le x'_{m,n}(t) \le M, \quad \text{for } t \in [0, n], \tag{3.9}$$

and $x_{m,n}$ satisfies

$$\begin{aligned} &x'' + \phi(t)f(t, x, x') = 0, \quad 0 < t < n, \ t \ne t_k, \\ &\Delta x'(t_k) = b_k x'(t_k), \qquad \Delta x(t_k) = a_k x(t_k), \quad k = 1, 2, \ldots \\ &x(0) = \alpha x(\eta), \qquad x'(n) = \frac{1}{m}. \end{aligned}$$

Now (H_5) guarantees the existence of a function $\psi_M(t)$ continuous on $[0, +\infty)$ and positive on $(0, +\infty)$, and a constant γ, $0 \le \gamma < 1$ with $f(t, x_{m,n}(t), x'_{m,n}(t)) \ge \psi_M(t)(x'_{m,n}(t))^\gamma$ for $(t, x_{m,n}(t), x'_{m,n}(t)) \in [0, n] \times [0, M]^2$. From

$$-x''_{m,n}(t) = \phi(t)f(t, x_{m,n}(t), x'_{m,n}(t)) \ge \phi(t)\psi_M(t)(x'_{m,n})^\gamma, \quad t \in [0, n],$$

and $x'_{m,n}(t_k^+) = (1 + b_k)x'_{m,n}(t_k)$, $x_{m,n}(t_k^+) = (1 + a_k)x_{m,n}(t_k)$, we have

$$x'_{m,n}(t) \ge \left((1-\gamma) \int_t^n \prod_{t<t_k\le s} (1+b_k)^{\gamma-1} \psi_M(s)\phi(s)\mathrm{d}s \right)^{1/(1-\gamma)}, \quad t \in [0, n],$$

and

$$x_{m,n}(t) \ge \int_0^t \prod_{s<t_k<t} (1+a_k) \left((1-\gamma) \int_s^n \prod_{s<t_k\le\tau} (1+b_k)^{\gamma-1} \psi_M(\tau)\phi(\tau)\mathrm{d}\tau \right)^{1/(1-\gamma)} \mathrm{d}s, \quad t \in [0, n].$$

It is easy to see that $\{x_{m,n}\}$, $m \in \{n_0, n_0 + 1, \ldots\}$ is a bounded, quasiequicontinuous family on $[0, n]$; Lemma 2.1 guarantees the existence of a subsequence N^0 of $\{n_0, n_0 + 1, \ldots, \}$ and a function $x_n \in PC^1[0, n]$ with $x_{m,n}$ converging uniformly on $[0, n]$ to x_n as $m \to +\infty$ through N^0 and

$$0 \le x_n(t) \le M, \qquad 0 \le x'_n(t) \le M, \tag{3.10}$$

$$x_n(t) \ge \int_0^t \prod_{s<t_k<t} (1+a_k) \left((1-\gamma) \int_s^n \prod_{s<t_k<\tau} (1+b_k)^{\gamma-1} \psi_M(\tau)\phi(\tau)\mathrm{d}\tau \right)^{1/(1-\gamma)} \mathrm{d}s, \tag{3.11}$$

and $x_n(t)$ is a solution of (3.1). Also, from (3.10) and (H_3), we have

$$0 \le -x_n''(t) \le \phi(t)\omega(M), \quad \text{for } t \in [0, n].$$

In addition we have

$$x_n'(t) \le \omega(M) \int_t^n \prod_{t<t_k\le s} (1+b_k^-)^{-1}\phi(s)\mathrm{d}s.$$

To show that (1.1) has solutions, we will apply a diagonalization argument. Let

$$y_n(t) = \begin{cases} x_n(t), & t \in [0, n] \\ x_n(n), & t \in [n, \infty). \end{cases}$$

Then by (3.10) we have $0 < y_n(t) \le M$, $0 < y_n'(t) \le M$. And for $t, s \in (t_k, t_{k+1}]$ it is to see that

$$|y_n'(t) - y_n'(s)| \le \omega(M) \left| \int_s^t \phi(\tau)\mathrm{d}\tau \right|.$$

In addition

$$y_n'(t) \le \omega(M) \int_t^\infty \prod_{t<t_k\le s} (1+b_k^-)^{-1}\phi(s)\mathrm{d}s,$$

and

$$y_n(t) \ge e_l(t), \quad \text{for } t \in [0, [\eta]+l], \tag{3.12}$$

where l is any positive integer with $[\eta]+l \le n$, e_l denoted by

$$e_l(t) = \int_0^t \prod_{s<t_k<t} (1+a_k) \left((1-\gamma) \int_s^{[\eta]+l} \prod_{s<t_k\le\tau} (1+b_k)^{\gamma-1}\psi_M(\tau)\phi(\tau)\mathrm{d}\tau \right)^{1/(1-\gamma)} \mathrm{d}s.$$

Also notice for $n \in N \setminus \{0\}$ that

$$y_n(t) \ge e_1(t), \quad \text{for } t \in [0, [\eta]+1]. \tag{3.13}$$

Lemma 2.1 guarantees the existence of a subsequence N^1 of $N \setminus \{0\}$ and a function $z_1 \in PC^1[0, [\eta]+1]$ with y_n converging uniformly on $[0, 1]$ to z_1 as $n \to +\infty$ through N^1. Also from (3.13), $z_1(t) \ge e_1(t)$ for $t \in [0, [\eta]+1]$.

Now notice from (3.12) that for $n \in N^1 \setminus \{[\eta]+1\}$,

$$y_n(t) \ge e_2(t) \quad \text{for } t \in [0, [\eta]+2].$$

Lemma 2.1 guarantees the existence of a subsequence N^2 of $N^1 \setminus \{[\eta]+1\}$ and a function $z_2 \in PC^1[0, [\eta]+2]$ with y_n converging uniformly on $[0, [\eta]+2]$ to z_2 as $n \to \infty$ through N^2. Also $z_2(t) \ge e_2(t)$ for $t \in [0, [\eta]+2]$ and $z_2 = z_1$ on $[0, [\eta]+1]$. Proceed inductively to obtain for $k = 1, 2, \ldots$ a subsequence N^k of $N^{k-1} \setminus \{[\eta]+k-1\}$ and a function $z_k \in PC^1[0, [\eta]+k]$ with y_n converging uniformly on $[0, [\eta]+k]$ to z_k as $n \to +\infty$ through N^k. Also $z_k(t) \ge e_k(t)$ for $t \in [0, [\eta]+k]$ and $z_k = z_{k-1}$ on $[0, [\eta]+k-1]$.

Define a function x as follows: Fix $t \in [0, +\infty)$ and let $k \in N \setminus \{0\}$ with $t \le [\eta]+k$. Define $x(t) = z_k(t)$. Note that z is well defined and $x(t) = z_k(t) > 0$.

Fix $t \in [0, +\infty)$ and choose $[\eta]+k \ge t$. Then for $n \in N^k$ we have

$$\begin{aligned} y_n(t) = {} & \left[\frac{\alpha\eta}{1-\alpha} + t\right] y_n'([\eta]+k) + \frac{\alpha}{1-\alpha} \int_0^\eta \int_s^{[\eta]+k} \phi(\tau) f(\tau, y_n(\tau), y_n'(\tau))\mathrm{d}\tau\mathrm{d}s \\ & - \frac{\alpha}{1-\alpha} \sum_{0<t_k<[\eta]+k} b_k y_n'(t_k)\eta + \frac{\alpha}{1-\alpha}\left[\sum_{0<t_i\le\eta} b_i y_n'(t_i)(\eta - t_i) + \sum_{0<t_i<\eta} a_i y_n(t_i) \right] \\ & + \int_0^t \int_s^{[\eta]+k} \phi(\tau) f(\tau, y_n(\tau), y_n'(\tau))\mathrm{d}\tau\mathrm{d}s - \sum_{0<t_i<[\eta]+k} b_i y_n'(t_i) t + \sum_{0<t_i\le t} b_i y_n'(t_i)(t-t_i) + \sum_{0<t_i<t} a_i y_n(t_i). \end{aligned}$$

Letting $n \to +\infty$ through N^k (notice that $n \ge [\eta]+k \ge t$) we have

$$\begin{aligned} z_k(t) = {} & \left[\frac{\alpha\eta}{1-\alpha} + t\right] z_k'([\eta]+k) + \frac{\alpha}{1-\alpha} \int_0^\eta \int_s^{[\eta]+k} \phi(\tau) f(\tau, z_k(\tau), z_k'(\tau))\mathrm{d}\tau\mathrm{d}s \\ & - \frac{\alpha}{1-\alpha} \sum_{0<t_k<[\eta]+k} b_k z_k'(t_k)\eta + \frac{\alpha}{1-\alpha}\left[\sum_{0<t_i\le\eta} b_i z_k'(t_i)(\eta - t_i) + \sum_{0<t_i<\eta} a_i z_k(t_i) \right] \end{aligned}$$

$$+\int_0^t\int_s^{[\eta]+k}\phi(\tau)f(\tau,z_k(\tau),z_k'(\tau))\mathrm{d}\tau\mathrm{d}s-\sum_{0<t_i<[\eta]+k}b_iz_k'(t_i)t+\sum_{0<t_i\le t}b_iz_k'(t_i)(t-t_i)+\sum_{0<t_i<t}a_iz_k(t_i).$$

Thus

$$\begin{aligned}x(t)&=\left[\frac{\alpha\eta}{1-\alpha}+t\right]x'([\eta]+k)+\frac{\alpha}{1-\alpha}\int_0^\eta\int_s^{[\eta]+k}\phi(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s\\&\quad-\frac{\alpha}{1-\alpha}\sum_{0<t_k<[\eta]+k}b_kx'(t_k)\eta+\frac{\alpha}{1-\alpha}\left[\sum_{0<t_i\le\eta}b_ix'(t_i)(\eta-t_i)+\sum_{0<t_i<\eta}a_ix(t_i)\right]\\&\quad+\int_0^t\int_s^{[\eta]+k}\phi(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s-\sum_{0<t_i<[\eta]+k}b_ix'(t_i)t+\sum_{0<t_i\le t}b_ix'(t_i)(t-t_i)+\sum_{0<t_i<t}a_ix(t_i).\end{aligned}$$

Consequently $x\in PC^2[0,+\infty)$ with $x''+\phi(t)f(t,x,x')=0$ for $0<t<\infty$, and $x(0)=\alpha x(\eta),\ x'(+\infty)=0$. Hence x is a solution of (1.1), which completes the proof. $\square$

4. An example

Consider the boundary value problem

$$\begin{cases}x''(t)+\phi(t)\left(\ln(1+|x|)+\mathrm{e}^{-t}\left(1+x^{\frac{1}{5}}x'^{\frac{1}{5}}\right)\right)=0,\quad 0<t<+\infty,t\ne t_k,\\ \Delta x'(t_k)=b_kx'(t_k),\qquad \Delta x(t_k)=a_kx(t_k),\quad k=1,2,\ldots,\\ x(0)=\dfrac{1}{2}x(5),\qquad x'(+\infty)=0,\end{cases}\tag{4.1}$$

where $\phi(t)=\mathrm{e}^{-\beta t},\ \beta>0,\ a_k=-\frac{1}{k+1},b_k=\frac{1}{2^k}$. By computation we have

$$Q_\infty=\int_0^{+\infty}\prod_{0<t_k<s}(1+b_k^-)^{-1}\phi(s)\mathrm{d}s=\frac{1}{\beta},\qquad R_\infty=\int_0^{+\infty}s\phi(s)\mathrm{d}s=\frac{1}{\beta^2},$$

$$A=\prod_{k=1}^\infty(1+a_k^+)=1<2,\qquad B=\prod_{k=1}^\infty(1+b_k^-)^{-1}=1,\qquad B_1=\prod_{k=1}^\infty(1+b_k^+)^{-1}=\prod_{k=1}^\infty\left(1+\frac{1}{2^k}\right)^{-1}<+\infty.$$

Letting

$$\omega(r)=\ln(1+r)+\left(1+r^{\frac{1}{5}}\right)r^{\frac{1}{5}},\quad r\in[0,+\infty),$$

$$\psi_H(t)=\phi(t)\mathrm{e}^{-t},\quad t\in[0,+\infty),\ H>0,$$

it follows that $\omega\ge 0$ is continuous and nondecreasing on $[0,+\infty)$, $\psi_H(t)\ge 0$ is continuous on $[0,+\infty)$, and for $(t,u,v)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty)$,

$$f(t,u,v)\le\omega(\max\{u,v\}),\quad f(t,u,v)\ge\psi_H(t)v^{\frac{1}{5}},$$

and

$$\lim_{r\to+\infty}\frac{r}{\left(\ln(1+r)+(1+r^{\frac{1}{5}})\right)r^{\frac{1}{5}}\max\left\{Q_\infty,\frac{\alpha A^2B}{1-\alpha A}\int_0^\eta s\phi(s)\mathrm{d}s+ABR_\infty\right\}}=+\infty.$$

Hence the conditions (H_1)–(H_5) of Theorem 3.1 hold, and so (4.1) has solutions $x\in PC^1[0,+\infty)\cap PC^2(0,+\infty)$ with $x>0$ on $(0,+\infty)$.

Acknowledgement

The authors thank the referee for helpful comments.

References

[1] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[2] D.D. Bainov, P.S. Simeonov, Impulsive Differential Equations: Periodic Solution and Applications, Longman Scientific and Technical, Harlow, 1993.
[3] S.T. Zavalishchin, A.N. Sesekin, Dynamic impulse systems, in: Theory and Applications, Kluwer Academic Publishers Group, Dordrecht, 1997.
[4] T.E. Carter, Necessary and sufficient conditions for optimal impulsive rendezvous with linear equations of motion, Dyn. Control 10 (2000) 219–227.
[5] X. Liu, A.R. Willms, Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft, Math. Probl. Eng. 2 (1996) 277–299.

[6] A.F.B.A. Prado, Bi-impulsive control to build a satellite constellation, Nonlinear Dyn. Syst. Theory 5 (2005) 169–175.
[7] J. Shen, J. Li, Existence and global attractivity of positive periodic solutions for impulsive predator–prey model with dispersion and time delays, Nonlinear Anal. 10 (2009) 227–243.
[8] S. Tang, J. Chen, Density-dependent birth rate, birth pulses and their population dynamic consequences, J. Math. Biol. 44 (2002) 185–199.
[9] X. Liu, D. Guo, Initial value problems for first order impulsive integro-differential equations in Banach spaces, Commun. Appl. Nonlinear Anal. 2 (1995) 65–83.
[10] C. Li, X. Liao, X. Yang, Impulsive stabilization and synchronization of a class of chaotic delay systems, Chaos 15 (2005) 043103. 9 pages.
[11] J. Li, J. Shen, Periodic boundary value problems for impulsive integro-differential equations, Appl. Math. Comput. 183 (2006) 890–902.
[12] J.J. Nieto, Impulsive resonance periodic problems of first order, Appl. Math. Lett. 15 (2002) 489–493.
[13] D. Guo, Existence of positive solutions for nth-order nonlinear impulsive singular integro-differential equations in Banach spaces, Nonlinear Anal. 68 (2008) 2727–2740.
[14] B. Yan, Boundary value problems on the half-line with impulses and infinite delay, J. Math. Anal. Appl. 259 (2001) 94–114.
[15] J. Li, J. Shen, Existence of positive solution for second-order impulsive boundary value problems on infinity intervals, Bound. Value Probl. (2006) Article ID 14594 11 pages.
[16] Ravi P. Agarwal, Donal O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic Publishers, 2001.